LESSON NOTE ON PERMUTATIONS AND COMBINATIONS

(WEEK 2 & 3)

CLASS: SS2

OBJECTIVES:

BY THE END OF THE LESSON, STUDENTS WILL BE ABLE TO:

1. STATE THE FUNDAMENTAL PRINCIPLE OF COUNTING;
2. DEFINE N! AND EVALUATE IT FOR DIFFERENT VALUES OF N;
3. STATE THAT PERMUTATION IS AN ARRANGEMENT AND WRITE THE MEANING OF ${}^{N}P_{R}$
4. STATE THAT ${}^{N}P_{R} = \frac{n!}{(n-r)!}$ AND APPLY TO SOLVE PROBLEMS
5. STATE THAT A COMBINATION IS A SELECTION AND WRITE THE MEANING OF ${}^{N}C_{R}$
6. DISTINGUISH BETWEEN PERMUTATIONS AND COMBINATIONS;
7. DERIVE ${}^{N}C_{R} = \frac{n!}{(n-r)!r!}$ AND APPLY THE RESULTS TO SOLVE PROBLEMS

EXPECTED BACKGROUND KNOWLEDGE

1. NUMBER SYSTEMS
2. FOUR FUNDAMENTAL OPERATIONS

INSTRUCTION

ALL EVALUATIONS SHOULD BE DONE AND SUBMITTED ON THE DATE GIVEN BY THE SCHOOL.

COUNTING PRINCIPLE

IF ANY EVENT CAN OCCUR IN *M* WAYS AND AFTER IT HAPPENS IN ANY ONE OF THESE WAYS, A SECOND EVENT CAN OCCUR IN *N* WAYS, THEN BOTH THE EVENTS TOGETHER CAN OCCUR IN *M*□*N* WAYS.

EXAMPLE 1:

HOW MANY MULTIPLES OF 5 ARE THERE FROM 10 TO 95?

SOLUTION:

AS YOU KNOW, MULTIPLES OF 5 ARE INTEGERS HAVING 0 OR 5 IN THE DIGIT TO THE EXTREME RIGHT (I.E. THE UNIT'S PLACE). THE FIRST DIGIT FROM THE RIGHT CAN BE CHOSEN IN 2 WAYS. THE SECOND DIGIT CAN BE ANY ONE OF 1, 2, 3, 4, 5, 6, 7, 8, 9. I.E. THERE ARE 9 CHOICES FOR THE SECOND DIGIT. THUS, THERE ARE 2□9 □18 MULTIPLES OF 5 FROM 10 TO 95.

EXAMPLE 2:

IN A CITY, THE BUS ROUTE NUMBERS CONSIST OF A NATURAL NUMBER LESS THAN 100, FOLLOWED BY ONE OF THE LETTERS *A, B, C, D, E* AND *F*. HOW MANY DIFFERENT BUS ROUTES ARE POSSIBLE?

SOLUTION: THE NUMBER CAN BE ANY ONE OF THE NATURAL NUMBERS FROM 1 TO 99.

THERE ARE 99 CHOICES FOR THE NUMBER.

THE LETTER CAN BE CHOSEN IN 6 WAYS.

□□NUMBER OF POSSIBLE BUS ROUTES ARE 99□6 □□594.

EXAMPLE 3:
THERE ARE 3 QUESTIONS IN A QUESTION PAPER. IF THE QUESTIONS HAVE 4, 3 AND 2 SOLUTIONS RESPECTIVELY, FIND THE TOTAL NUMBER OF SOLUTIONS.

SOLUTION: HERE QUESTION 1 HAS 4 SOLUTIONS, QUESTION 2 HAS 3 SOLUTIONS AND QUESTION 3 HAS 2 SOLUTIONS.
□□BY THE MULTIPLICATION (COUNTING) RULE, TOTAL NUMBER OF SOLUTIONS =
4□3□2 = 24

EXAMPLE 4:
CONSIDER THE WORD ROTOR. WHICHEVER WAY YOU READ IT, FROM LEFT TO RIGHT OR FROM RIGHT TO LEFT, YOU GET THE SAME WORD. SUCH A WORD IS KNOWN AS *PALINDROME*. FIND THE MAXIMUM POSSIBLE NUMBER OF 5-LETTER PALINDROMES.

SOLUTION:
THE FIRST LETTER FROM THE RIGHT CAN BE CHOSEN IN 26 WAYS BECAUSE THERE ARE 26 ALPHABETS. HAVING CHOSEN THIS, THE SECOND LETTER CAN BE CHOSEN IN 26 WAYS
□□THE FIRST TWO LETTERS CAN BE CHOSEN IN 26□26 □□676 WAYS
HAVING CHOSEN THE FIRST TWO LETTERS, THE THIRD LETTER CAN BE CHOSEN IN 26 WAYS.
□ALL THE THREE LETTERS CAN BE CHOSEN IN 676□26 □17576 WAYS.
IT IMPLIES THAT THE MAXIMUM POSSIBLE NUMBER OF FIVE LETTER PALINDROMES IS 17576 BECAUSE THE FOURTH LETTER IS THE SAME AS THE SECOND LETTER AND THE FIFTH LETTER IS THE SAME AS THE FIRST LETTER.

EXAMPLE 5:
HOW MANY 3-DIGIT NUMBERS CAN BE FORMED WITH THE DIGITS 1, 4, 7, 8 AND 9 IF THE DIGITS ARE NOT REPEATED?

SOLUTION:
THREE DIGIT NUMBER WILL HAVE UNITS, TENS AND HUNDREDS PLACE. OUT OF 5 GIVEN DIGITS ANY ONE CAN TAKE THE UNITS PLACE. THIS CAN BE DONE IN 5 WAYS. ... (*I*)
AFTER FILLING THE UNITS PLACE, ANY OF THE FOUR REMAINING DIGITS CAN TAKE THE TENS PLACE. THIS CAN BE DONE IN 4 WAYS. ... (*II*)
AFTER FILLING IN TEN'S PLACE, HUNDREDS PLACE CAN BE FILLED FROM ANY OF THE THREE REMAINING DIGITS. THIS CAN BE DONE IN 3 WAYS. ... (*III*)
□□BY COUNTING PRINCIPLE, THE NUMBER OF 3 DIGIT NUMBERS = 5□4□3 = 60

IF THERE ARE *N* EVENTS AND IF THE FIRST EVENT CAN OCCUR IN m_1 WAYS, THE SECOND EVENT CAN OCCUR IN m_2 WAYS AFTER THE FIRST EVENT HAS OCCURRED, THE THIRD EVENT CAN OCCUR IN m_3 WAYS AFTER THE SECOND EVENT HAS OCCURRED, AND SO ON, THEN ALL THE *N* EVENTS CAN OCCUR IN $m_1 \times m_2 \times \ldots \times m_{n-1} \times m_n$□WAYS.

EXAMPLE 6:
SUPPOSE YOU CAN TRAVEL FROM A PLACE *A* TO A PLACE *B* BY 3 BUSES, FROM PLACE *B* TO PLACE *C* BY 4 BUSES, FROM PLACE *C* TO PLACE *D* BY 2 BUSES AND FROM PLACE *D* TO PLACE *E* BY 3 BUSES. IN HOW MANY WAYS CAN YOU TRAVEL FROM *A* TO *E*?

SOLUTION:
THE BUS FROM *A* TO *B* CAN BE SELECTED IN 3 WAYS. THE BUS FROM *B* TO *C* CAN BE SELECTED IN 4 WAYS. THE BUS FROM *C* TO *D* CAN BE SELECTED IN 2 WAYS. THE BUS FROM *D* TO *E* CAN BE SELECTED IN 3 WAYS.
SO, BY THE GENERAL COUNTING PRINCIPLE, ONE CAN TRAVEL FROM *A* TO *E* IN
3□4□2□3 WAYS = 72 WAYS.

PERMUTATIONS
BY PERMUTATION WE MEAN AN ARRANGEMENT OF OBJECTS IN A PARTICULAR ORDER.

THEOREM: THE TOTAL NUMBER OF PERMUTATIONS OF *N* OBJECTS IS
$n \times (n - 1) \times \ldots \times 2 \times 1$

THE PRODUCT $n \times (n - 1) \times \ldots \times 2 \times 1$ OCCURS SO OFTEN IN MATHEMATICS THAT IT DESERVES A NAME AND NOTATION. IT IS USUALLY DENOTED BY ***N*!** **(**READ AS ***N* FACTORIAL**).
$N! = n \times (n - 1) \times \ldots \times 2 \times 1$

EXAMPLE 7:
EVALUATE (A) 3! (B) 2! + 4! (C) 2!□3!
SOLUTION:
(a) 3! □□3□2□1 □□6
(b) 2! □□2□1 □□2
4! □□4□3□□2□1 □□24
Therefore, 2! □□4! □□2 □□24 □□26
(c) 2! □□3! □□2□□6 □□12
Notice that *n*! satisfies the relation *n*! □□*n*□□*n* □1□□!
This is because,
n (*n* – 1)! = *n* [(*n* – 1) . (*n* – 2)... 2.1] = *n* . (*n* – 1) . (*n* – 2) ...2.1= *n*!
Of course, the above relation is valid only for *n* □□2 because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define 0! = 1 then the relation holds for *n* = 1 also.

Example 8:
Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution:
We have to arrange 6 books. The number of permutations of *n* objects is *n*! = *n*. (*n* – 1) . (*n* – 2) ... 2.1

Here n = 6 and therefore, number of permutations is 6.5.4.3.2.1 = 720

PERMUTATION OF r OBJECTS OUT OF n OBJECTS

The number of permutations of r objects out of n objects is $n \times (n-1) \times ... \times (n - r + 1)$. The number of permutations of r objects out of n objects is usually denoted by ${}^{n}P_{r} = \frac{n!}{(n-r)!}$

Example 9:

Evaluate: (a) ${}^{4}P_{2}$ (b) ${}^{6}P_{3}$ (c) ${}^{6}P_{3} \times {}^{5}P_{2}$ (d) ${}^{4}P_{3} / {}^{3}P_{2}$

Solution:

(a) ${}^{4}P_{2} = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4\times3\times2\times1}{2\times1} = 12$

(b) ${}^{6}P_{3} = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1} = 120$

(c) ${}^{5}P_{2} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3\times2\times1}{3\times2\times1} = 20$

So ${}^{6}P_{3} \times {}^{5}P_{2} = 120 \times 20 = 2400$

(d) ${}^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4\times3\times2\times1}{1} = 24$

${}^{3}P_{2} = 6$

So, ${}^{4}P_{3} / {}^{3}P_{2} = 24/6 = 4$

Example 10:

If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution: We have to find number of permutations of 4 objects out of 6 objects.

This number is

${}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 360$

Example 11:

Find the value of ${}^{n}P_{0}$

Solution: Here r = 0 so, ${}^{n}P_{0} = \frac{n!}{n!} = 1$

PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 12:

Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution: Let the beds be numbered 1 to 7.

Case 1: Suppose Anju is allotted bed number 1. Then, Parvin cannot be allotted bed number 2. So Parvin can be allotted a bed in 5 ways. After allotting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways. So, in this case the beds can be allotted in 5□5! *ways* □□600 ways.

Case 2: Anju is allotted bed number 7. Then, Parvin cannot be allotted bed number 6 As in Case 1, the beds can be allotted in 600 ways.

Case 3: Anju is allotted one of the beds numbered 2,3,4,5 or 6. Parvin cannot be allotted the beds on the right hand side and left hand side of Anju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin. Therefore, Parvin can be allotted a bed in 4 ways in all these cases. After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways. Therefore, in each of these cases, the beds can be allotted in 4□□5! □□480 ways.
□□The beds can be allotted in
(2□600 □□5□□480) *ways* □□(1200 □□2400)*ways* □□3600 ways.

Example 13:
In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution:
They have to be arranged in the following way:
L T L T L T L T L
The 5 lions should be arranged in the 5 places marked 'L'.
This can be done in 5! ways.
The 4 tigers should be in the 4 places marked 'T'.
This can be done in 4! ways.
Therefore, the lions and the tigers can be arranged in 5!□□4! *ways* □□2880 ways.

Example 14:
There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairy tales, novels and plays.

Solution:
There are 4 books on fairy tales and they have to be put together.
They can be arranged in 4! ways.
Similarly, there are 5 novels.
They can be arranged in 5! ways.
And there are 3 plays.
They can be arranged in 3! ways.
So, by the counting principle all of them together can be arranged in 4!□5!□3! ways = 17280 ways.

Example 15:
Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 14. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution:
First, we consider the books on fairy tales, novels and plays as single objects. These three objects can be arranged in 3! ways = 6 ways.
Let us fix one of these 6 arrangements.
This may give us a specific order, say, novels □□fairy tales □□plays.
Given this order, the books on the same subject can be arranged as follows.
The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 5 novels can be arranged in 5! = 120 ways.
The 3 plays can be arranged in 3! = 6 ways.
For a given order, the books can be arranged in 24□120□6 □17280 ways.
Therefore, for all the 6 possible orders the books can be arranged in 6□17280 □103680 ways.

Example 16:
In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution:
Let 4 girls be one unit and now there are 6 units in all.
They can be arranged in 6! ways.
In each of these arrangements 4 girls can be arranged in 4! ways.
□□Total number of arrangements in which girls are always together
= 6! □□4! = 720 □□24 = 17280

COMBINATIONS
Let n □1 be an integer and r □□n. Let us denote the number of ways of choosing r objects out of n objects by ${}^{n}C_{r}$. Then

${}^{n}C_{r} = {}^{n}P_{r} \div r! = \frac{n!}{(n-r)!r!}$

Example 17: Evaluate each of the following
(a) ${}^{5}C_{2}$ (b) ${}^{5}C_{3}$ (c) ${}^{4}C_{3} + {}^{4}C_{2}$ (d) ${}^{6}C_{3} \div {}^{4}C_{2}$

Solution:
(a) ${}^{5}C_{2} = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 10$

(b) ${}^{5}C_{3} = \frac{5!}{(5-3)!3!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{2\times1\times3\times2\times1} = 10$

(c) ${}^{4}C_{3} = \frac{4!}{(4-3)!3!} = \frac{4!}{1!3!} = \frac{4\times3\times2\times1}{1\times3\times2\times1} = 4$

${}^{4}C_{2} = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4\times3\times2\times1}{2\times1\times2\times1} = 6$

Therefore ${}^{4}C_{3} + {}^{4}C_{2} = 4 + 6 = 10$

(d) ${}^{6}C_{3} = \frac{6!}{(6-3)!3!} = \frac{6!}{3!3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1} = 20$

Therefore, ${}^{6}C_{3} \div {}^{4}C_{2} = \frac{20}{6} = \frac{10}{3}$

Example 18:
Find the number of subsets of the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} having 4 elements.

Solution: Here the order of choosing the elements doesn't matter and this is a problem in combinations. We have to find the number of ways of choosing 4 elements of this set which has 11 elements. This can be done in

$$^{11}C_4 = \frac{11!}{(11-4)!4!} = \frac{11!}{7!4!} = \frac{11\times10\times9\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1\times4\times3\times2\times1} = 330$$

Example 19:
12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution: For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $^{12}C_4$ □□□□□. Therefore, we can draw 495 quadrilaterals.

Example 20:
In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution: Number of ways of choosing 2 black pens from 5 black pens is
$^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 10$
Number of ways of choosing 2 white pens from 3 white pens is
$^3C_2 = 3$
Number of ways of choosing 2 red pens from 4 red pens is
$^4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4\times3\times2\times1}{2\times1\times2\times1} = 6$
□□By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in 10□3□6 □180 ways.

Example 21:
A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when
(i) at least 2 women are included?
(ii) at most 2 women are included?

Solution:
(i) When at least 2 women are included. The committee may consist of
3 women, 2 men: It can be done in $^4C_3 \times {}^6C_2 = 60$ ways
or
4 women, 1 man: It can be done in $^4C_4 \times {}^6C_1 = 6$ ways
or
2 women, 3 men: It can be done in $^4C_2 \times {}^6C_3 = 120$ ways
□□Total number of ways of forming the committee is
□120 □□60 □□6 □186

(ii) When at most 2 women are included the committee may consist of
2 women, 3 men: It can be done in $^4C_2 \times {}^6C_3 = 120$ ways
or
1 woman, 4 men: It can be done in $^4C_1 \times {}^6C_4 = 60$ ways
or
5 men: It can be done in $^6C_5 = 6$ ways
□□Total number of ways of forming the committee is
120 + 60 + 6 = 186

SOME PROPERTIES OF COMBINATION

1. $^nC_0 = 1$

2. ${}^{n}C_r = {}^{n}C_{n-r}$. This also means that if ${}^{n}C_r = {}^{n}C_k$ then, r+k=n and vice versa.

Example 22:
If ${}^{n}C_{10} = {}^{n}C_{12}$ find n.

Solution:
Using ${}^{n}C_r = {}^{n}C_k$ then, r + k = n
So, 10 + 12 = 22

PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS
So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 23:
There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf?

Solution: 4 novels can be selected out of 5 in ${}^{5}C_4 = 5$.
2 biographies can be selected out of 4 in ${}^{4}C_2 = 6$.
Number of ways of arranging novels and biographies is $5 \times 6 = 30$
After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6! = 720 ways.
By the Counting Principle, the total number of arrangements is 30□720 □□21600

Example 24:
From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels?

Solution:
From 5 consonants, 3 consonants can be selected in ${}^{5}C_3 = 10$ ways.
From 4 vowels, 2 vowels can be selected in ${}^{4}C_2 = 6$ ways.
Now with every selection, number of ways of arranging 5 letters is
${}^{5}P_5 = 5! = 120$ ways.
□□Total number of words is $10 \times 6 \times 120 = 7200$.

SUMMARY:
1. Fundamental principle of counting states that if there are *n* events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the*n* events can occur in m_1 □□m_2 □□m_3 □□... □□m_{n-1} □□m_n□ways.
2. The number of permutations of *n* objects taken all at a time is *n*!
3. ${}^{n}P_r = \frac{n!}{(n-r)!}$
4. ${}^{n}P_n = n!$
5. The number of ways of selecting *r* objects out of *n* objects is
   ${}^{n}C_r = \frac{n!}{(n-r)!r!}$
6. ${}^{n}C_r = {}^{n}C_{n-r}$
7. ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

EVALUATION AND ASSIGNMENT

1. There are 8 true - false questions in an examination. How many responses are possible?
2. The six faces of a die are numbered 1, 2, 3, 4, 5 and 6. Two such dice are thrown simultaneously. In how many ways can they turn up?
3. A restaurant has 3 vegetables, 2 salads and 2 types of bread. If a customer wants 1 vegetable, 1 salad and 1 bread, how many choices does he have?
4. Suppose you want to paper your walls. Wall papers are available in 4 different backgrounds colours with 7 different designs of 5 different colours on them. In how many ways can you select your wall paper?
5. In how many ways can 7 students be seated in a row on 7 seats?
6. Determine the number of 8 letter words that can be formed from the letters of the word *ALTRUISM*.
7. Determine the maximum number of 3- letter words that can be formed from the letters of the word *POLICY*.
8. In how many ways can you arrange the letters of the word *ATTAIN* so that the *T*s are together?
9. Suppose that you own a shop which sells televisions. You are selling 5 different kinds of television sets, but your show case has enough space for display of 3 television sets only. In how many ways can you select the television sets for the display?
10. In how many ways can a committee of 9 can be selected from a group of 13?
11. In how ways can 6 persons be selected from 4 grade 1 and 7 grade II officers, so as to include at least two officers from each category?
12. Out of 6 boys and 4 girls, a committee of 5 has to be formed. In how many ways can this be done if we take:

(a) 2 girls.
(b) at least 2 girls.

13. The English alphabet has 5 vowels and 21 consonants. What is the maximum number of words that can be formed from the alphabet with 2 different vowels and 2 different consonants?
14. From 5 consonants and 5 vowels, how many words can be formed using 3 consonants and 2 vowels?
15. Study about cyclic permutation and provide its formula.